

AN INVOLUTION BASED LEFT IDEAL IN THE HECKE ALGEBRA

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INTRODUCTION

0.1. Let W be a Coxeter group with set of simple reflections S and with length function $l : W \rightarrow \mathbf{N}$. Let u be an indeterminate. Let \mathfrak{H} be the $\mathbf{Q}(u)$ -vector space with basis $\{T_w; w \in W\}$. We regard \mathfrak{H} as an associative $\mathbf{Q}(u)$ -algebra (with 1) with multiplication defined by $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$, $(T_s + 1)(T_s - u^2) = 0$ if $s \in S$. Let $*$: $W \rightarrow W$ (or $w \mapsto w^*$) be an automorphism of W such that $S^* = S$, $*^2 = 1$. Let $\mathbf{I}_* = \{w \in W; w^* = w^{-1}\}$ be the set of "twisted involutions" of W . Let M be the $\mathbf{Q}(u)$ -vector space with basis $\{a_w; w \in \mathbf{I}_*\}$. Following [LV], for any $s \in S$ we define a $\mathbf{Q}(u)$ -linear map $T_s : M \rightarrow M$ by

$$\begin{aligned} T_s a_w &= u a_w + (u + 1) a_{sw} \text{ if } sw = ws^* > w; \\ T_s a_w &= (u^2 - u - 1) a_w + (u^2 - u) a_{sw} \text{ if } sw = ws^* < w; \\ T_s a_w &= a_{sws^*} \text{ if } sw \neq ws^* > w; \\ T_s a_w &= (u^2 - 1) a_w + u^2 a_{sws^*} \text{ if } sw \neq ws^* < w. \end{aligned}$$

(For x, y in W such that $y^{-1}x \in S$ or $xy^{-1} \in S$ we write $x < y$ or $y > x$ instead of $l(x) = l(y) - 1$.) According to [LV] and [L5], these linear maps define an \mathfrak{H} -module structure on M . Let $\hat{\mathfrak{H}}$ be the vector space consisting of all formal (possibly infinite) sums $\sum_{x \in W} c_x T_x$ where $c_x \in \mathbf{Q}(u)$. We can view \mathfrak{H} as a subspace of $\hat{\mathfrak{H}}$ in an obvious way. The \mathfrak{H} -module structure on \mathfrak{H} (left multiplication) extends in an obvious way to an \mathfrak{H} -module structure on $\hat{\mathfrak{H}}$. We set

$$X = \sum_{x \in W; x^* = x} u^{-l(x)} T_x \in \hat{\mathfrak{H}}.$$

The following is the main result of this paper:

Theorem 0.2. (a) *There exists a unique \mathfrak{H} -linear map $\mu : M \xrightarrow{\sim} \hat{\mathfrak{H}}$ such that $\mu(a_1) = X$. Moreover, μ is an isomorphism of M onto the \mathfrak{H} -submodule of $\hat{\mathfrak{H}}$ generated by X .*

(b) *Let $z \in \mathbf{I}_*$; we set $\mu(a_z) = \sum_{x \in W} N_z^x T_x$ where $N_z^x \in \mathbf{Q}(u)$. For any $x \in W$ we have $N_z^x \in \mathbf{Z}[u^{-1}]$, hence we can define $n_z^x = N_z^x|_{u^{-1}=0} \in \mathbf{Z}$.*

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(c) *There is a unique surjective function $\pi : W \rightarrow \mathbf{I}_*$ such that for $x \in W$, $z \in \mathbf{I}_*$ we have $n_z^x = 1$ if $z = \pi(x)$, $n_z^x = 0$ if $z \neq \pi(x)$. (Note that $\pi(1) = 1$.)*

This was conjectured in [L4, 3.4, 3.7] where it was verified for several W of low rank. In the case where W is of type A_n and $* = 1$, part (a) of the theorem was first proved by Hu and Zhang [HZ]. The proof of the theorem is given in Section 1. In Section 2 we will discuss a special case of Theorem 0.2. Section 3 is preparatory for Section 4. In Section 4 we discuss some applications of Theorem 0.2. For example, we show that if W is a Weyl group of type A_n and if E is an irreducible representation of \mathfrak{H} , then the action of X on E is through an operator of rank 1; in particular the image of this operator is a canonical line in E . As another application we show that if W is a Weyl group of classical type and E is an irreducible special representation of the asymptotic Hecke algebra attached to W then E admits a basis such that any canonical basis element of that algebra acts in this basis through a matrix with all entries in \mathbf{N} . A third application is a definition of a canonical $G(\mathbf{F}_q)$ -stable subspace \mathcal{F}' of the space of functions \mathcal{F} on the flag manifold of a Chevalley group $G(\mathbf{F}_q)$ over a finite field \mathbf{F}_q with the following properties: if $G = SL_n$, then \mathcal{F}' contains exactly one copy of each irreducible representation of $G(\mathbf{F}_q)$ which appears in \mathcal{F} ; in general, the dimension of \mathcal{F}' is a polynomial in q with coefficients in \mathbf{N} whose value at 1 is the number of involutions in W . This polynomial is the sum of the fake degrees of the various irreducible representations of the Hecke algebra (each one taken once).

1. PROOF OF THEOREM 0.2

1.1. The $\mathbf{Z}[u]$ -submodule of M with basis $\{a_w; w \in \mathbf{I}_*\}$ is stable under the maps $T_s : M \rightarrow M$ ($s \in S$) hence is stable under the action of T_x ($x \in W$) since T_x is a composition of various T_s . Hence for $x \in W$ we can write uniquely

$$T_x a_1 = \sum_{z \in \mathbf{I}_*} L_z^x a_z$$

where $L_z^x \in \mathbf{Z}[u]$.

1.2. For $x \in W, z \in \mathbf{I}_*, s \in S$ we show:

$$\begin{aligned} (u^2 - u)L_{sz}^x &= u^2 L_z^{sx} + (u^2 - u - 1)L_z^x \text{ if } sz = zs^* > z, sx < x; \\ (u + 1)L_{sz}^x - uL_z^x &= u^2 L_z^{sx} \text{ if } sz = zs^* < z, sx < x; \\ u^2 L_{szs^*}^x &= u^2 L_z^{sx} + (u^2 - 1)L_z^x \text{ if } sz \neq zs^* > z, sx < x; \\ L_{szs^*}^x &= u^2 L_z^{sx} \text{ if } sz \neq zs^* < z, sx < x; \\ uL_z^x + (u^2 - u)L_{sz}^x &= L_z^{sx} \text{ if } sz = zs^* > z, sx > x; \\ (u + 1)L_{sz}^x + (u^2 - u - 1)L_z^x &= L_z^{sx} \text{ if } sz = zs^* < z, sx > x; \\ u^2 L_{szs^*}^x &= L_z^{sx} \text{ if } sz \neq zs^* > z, sx > x; \\ L_{szs^*}^x + (u^2 - 1)L_z^x &= L_z^{sx} \text{ if } sz \neq zs^* < z, sx > x. \end{aligned}$$

1.3. For $x \in W, s \in S$ we have $T_s T_x a_1 = \sum_{z \in \mathbf{I}_*} L_z^x T_s a_z$. Note that $T_s T_x a_1 = T_{sx} a_1$ if $sx > x$ and $T_s T_x a_1 = u^2 T_{sx} a_1 + (u^2 - 1) T_x a_1$ if $sx < x$. Thus,

$$\begin{aligned} \sum_{z \in \mathbf{I}_*} L_z^x T_s a_z &= \sum_{z \in \mathbf{I}_*} L_z^{sx} a_z \text{ if } sx > x, \\ \sum_{z \in \mathbf{I}_*} L_z^x T_s a_z &= \sum_{z \in \mathbf{I}_*} u^2 L_z^{sx} a_z + \sum_{z \in \mathbf{I}_*} (u^2 - 1) L_z^x a_z \text{ if } sx < x. \end{aligned}$$

Using the formulas for $T_s a_z$ in 0.1 we see that

$$\begin{aligned} & \sum_{z \in \mathbf{I}_*; sz = zs^*, sz > z} L_z^x (ua_z + (u+1)a_{sz}) \\ & + \sum_{z \in \mathbf{I}_*; sz = zs^*, sz < z} L_z^x ((u^2 - u - 1)a_z + (u^2 - u)a_{sz}) \\ & + \sum_{z \in \mathbf{I}_*; sz \neq zs^*, sz > z} L_z^x a_{s_z s^*} + \sum_{z \in \mathbf{I}_*; sz \neq zs^*, sz < z} L_z^x ((u^2 - 1)a_z + u^2 a_{s_z s^*}) \end{aligned}$$

or equivalently

$$\begin{aligned} & \sum_{z \in \mathbf{I}_*; sz = zs^*, sz > z} u L_z^x a_z + \sum_{z \in \mathbf{I}_*; sz = zs^*, sz < z} (u+1) L_{s_z}^x a_z \\ & + \sum_{z \in \mathbf{I}_*; sz = zs^*, sz < z} (u^2 - u - 1) L_z^x a_z + \sum_{z \in \mathbf{I}_*; sz = zs^*, sz > z} (u^2 - u) L_{s_z}^x a_z \\ & + \sum_{z \in \mathbf{I}_*; sz \neq zs^*, sz < z} L_{s_z s^*}^x a_z + \sum_{z \in \mathbf{I}_*; sz \neq zs^*, sz < z} (u^2 - 1) L_z^x a_z \\ & + \sum_{z \in \mathbf{I}_*; sz \neq zs^*, sz > z} u^2 L_{s_z s^*}^x a_z \end{aligned}$$

is equal to

$$\begin{aligned} & \sum_{z \in \mathbf{I}_*} L_z^{sx} a_z \text{ if } sx > x, \\ & \text{or to} \\ & \sum_{z \in \mathbf{I}_*} u^2 L_z^{sx} a_z + \sum_{z \in \mathbf{I}_*} (u^2 - 1) L_z^x a_z \text{ if } sx < x. \end{aligned}$$

We now take the coefficients of a_z in the two sides of this equality. We obtain the equalities in 1.2.

1.4. Let $\bar{\cdot}: \mathbf{Z}[u, u^{-1}] \rightarrow \mathbf{Z}[u, u^{-1}]$ be the ring involution such that $\overline{u^n} = (-u)^{-n}$ for any $n \in \mathbf{Z}$. We apply $\bar{\cdot}$ to the equalities in 1.2 and multiply the resulting equalities by u^2 . We obtain the following equalities.

$$\begin{aligned} (1+u)\bar{L}_{s_z}^x &= \bar{L}_z^{sx} + (-u^2 + u + 1)\bar{L}_z^x \text{ if } sz = zs^* > z, sx < x; \\ (u^2 - u)\bar{L}_{s_z}^x + u\bar{L}_z^x &= \bar{L}_z^{sx} \text{ if } sz = zs^* < z, sx < x; \\ \bar{L}_{s_z s^*}^x &= \bar{L}_z^{sx} + (1 - u^2)\bar{L}_z^x \text{ if } sz \neq zs^* > z, sx < x; \\ u^2 \bar{L}_{s_z s^*}^x &= \bar{L}_z^{sx} \text{ if } sz \neq zs^* < z, sx < x; \\ -u\bar{L}_z^x + (1+u)\bar{L}_{s_z}^x &= u^2 \bar{L}_z^{sx} \text{ if } sz = zs^* > z, sx > x; \\ (u^2 - u)\bar{L}_{s_z}^x + (-u^2 + u + 1)\bar{L}_z^x &= u^2 \bar{L}_z^{sx} \text{ if } sz = zs^* < z, sx > x; \\ \bar{L}_{s_z s^*}^x &= u^2 \bar{L}_z^{sx} \text{ if } sz \neq zs^* > z, sx > x; \\ u^2 \bar{L}_{s_z s^*}^x + (1 - u^2)\bar{L}_z^x &= u^2 \bar{L}_z^{sx} \text{ if } sz \neq zs^* < z, sx > x. \end{aligned}$$

1.5. It is well known (see for example [L5]) that there is a unique function $\phi : \mathbf{I}_* \rightarrow \mathbf{N}$ such that $\phi(1) = 0$ and such that for any $z \in \mathbf{I}_*$ and any $s \in S$ such that $sz < z$ we have $\phi(z) = \phi(sz) + 1$ if $sz = zs^*$ and $\phi(z) = \phi(szs^*)$ if $sz \neq zs^*$. By induction on $l(z)$ we see that $\phi(z) = l(z) \pmod 2$ for any $z \in \mathbf{I}_*$. Hence for $z \in \mathbf{I}_*$ we can set $\epsilon(z) = (-1)^{(l(z)+\phi(z))/2}$. From the definitions we see that for any $z \in \mathbf{I}_*$ and any $s \in S$ we have

$$(a) \quad \epsilon(z) = -\epsilon(sz) \text{ if } sz = zs^* \text{ and } \epsilon(z) = -\epsilon(szs^*) \text{ if } sz \neq zs^*.$$

1.6. For $x \in W, z \in \mathbf{I}_*$ we set

$$\tilde{L}_z^x = (-1)^{l(x)} \epsilon(z) \bar{L}_z^x.$$

With this notation the formulas in 1.4 can be rewritten as follows.

$$\begin{aligned} (1+u)\tilde{L}_{sz}^x &= \tilde{L}_z^{sx} + (u^2 - u - 1)\tilde{L}_z^x \text{ if } sz = zs^* > z, sx < x; \\ (u^2 - u)\tilde{L}_{sz}^x - u\tilde{L}_z^x &= \tilde{L}_z^{sx} \text{ if } sz = zs^* < z, sx < x; \\ \tilde{L}_{szs^*}^x &= \tilde{L}_z^{sx} + (u^2 - 1)\tilde{L}_z^x \text{ if } sz \neq zs^* > z, sx < x; \\ u^2\tilde{L}_{szs^*}^x &= \tilde{L}_z^{sx} \text{ if } sz \neq zs^* < z, sx < x; \\ u\tilde{L}_z^x + (1+u)\tilde{L}_{sz}^x &= u^2\tilde{L}_z^{sx} \text{ if } sz = zs^* > z, sx > x; \\ (u^2 - u)\tilde{L}_{sz}^x + (u^2 - u - 1)\tilde{L}_z^x &= u^2\tilde{L}_z^{sx} \text{ if } sz = zs^* < z, sx > x; \\ \tilde{L}_{szs^*}^x &= u^2\tilde{L}_z^{sx} \text{ if } sz \neq zs^* > z, sx > x; \\ u^2\tilde{L}_{szs^*}^x + (u^2 - 1)\tilde{L}_z^x &= u^2\tilde{L}_z^{sx} \text{ if } sz \neq zs^* < z, sx > x. \end{aligned}$$

1.7. Giving an \mathfrak{H} -linear map $\mu : M \rightarrow \hat{\mathfrak{H}}$ is the same as giving a family of elements $Y_z \in \hat{\mathfrak{H}}$ (one for each $z \in \mathbf{I}_*$) such that for any $z \in \mathbf{I}_*, s \in S$ we have

$$\begin{aligned} T_s Y_z &= u Y_z + (u+1) Y_{sz} \text{ if } sz = zs^* > z; \\ T_s Y_z &= (u^2 - u - 1) Y_z + (u^2 - u) Y_{sz} \text{ if } sz = zs^* < z; \\ T_s Y_z &= Y_{szs^*} \text{ if } sz \neq zs^* > z; \\ T_s Y_z &= (u^2 - 1) Y_z + u^2 Y_{szs^*} \text{ if } sz \neq zs^* < z. \end{aligned}$$

Indeed, if μ is given then the elements $Y_z = \mu(a_z)$ satisfy the equations above. Conversely, if the elements Y_z are given as above then we can define a $\mathbf{Q}(u)$ -linear map $\mu : M \rightarrow \hat{\mathfrak{H}}$ by $\mu(a_z) = Y_z$ for all $z \in \mathbf{I}_*$. This map will be compatible with the action of T_s for any $s \in S$ hence it will be automatically \mathfrak{H} -linear. Setting $Y_z = \sum_{x \in W} N_z^x T_x$ where $N_z^x \in \mathbf{Q}(u)$ we see that giving an \mathfrak{H} -linear map $\mu : M \rightarrow \hat{\mathfrak{H}}$ is the same as giving a family of elements $\{N_z^x; (x, z) \in W \times \mathbf{I}_*\}$ in $\mathbf{Q}(u)$ such that the following equations are satisfied for any $z \in \mathbf{I}_*, s \in S$:

$$\begin{aligned} \sum_{x \in W} N_z^x T_s T_x &= \sum_{x \in W} u N_z^x T_x + \sum_{x \in W} (u+1) N_{sz}^x T_x \text{ if } sz = zs^* > z; \\ \sum_{x \in W} N_z^x T_s T_x &= \sum_{x \in W} (u^2 - u - 1) N_z^x T_x + \sum_{x \in W} (u^2 - u) N_{sz}^x T_x \text{ if } sz = zs^* < z; \\ \sum_{x \in W} N_z^x T_s T_x &= \sum_{x \in W} N_{szs^*}^x T_x \text{ if } sz \neq zs^* > z; \\ \sum_{x \in W} N_z^x T_s T_x &= \sum_{x \in W} (u^2 - 1) N_z^x T_x + \sum_{x \in W} u^2 N_{szs^*}^x T_x \text{ if } sz \neq zs^* < z. \end{aligned}$$

(We then say that the family $\{N_z^x; (x, z) \in W \times \mathbf{I}_*\}$ is admissible.) Here we replace

$$\begin{aligned}
\sum_{x \in W} N_z^x T_s T_x &= \sum_{x \in W; sx > x} N_z^x T_{sx} + \sum_{x \in W; sx < x} u^2 N_z^x T_{sx} + \sum_{x \in W; sx < x} (u^2 - 1) N_z^x T_x \\
&= \sum_{x \in W; sx < x} N_z^{sx} T_x + \sum_{x \in W; sx > x} u^2 N_z^{sx} T_x + \sum_{x \in W; sx < x} (u^2 - 1) N_z^x T_x \\
&= \sum_{x \in W; sx < x} (N_z^{sx} + (u^2 - 1) N_z^x) T_x + \sum_{x \in W; sx > x} u^2 N_z^{sx} T_x.
\end{aligned}$$

We see that the condition that $\{N_z^x; (x, z) \in W \times \mathbf{I}_*\}$ is admissible is equivalent to the following set of equations (with $x \in W, z \in \mathbf{I}_*, s \in S$).

$$\begin{aligned}
(1 + u) N_{sz}^x &= N_z^{sx} + (u^2 - u - 1) N_z^x \text{ if } sz = zs^* > z, sx < x; \\
(u^2 - u) N_{sz}^x - u N_z^x &= N_z^{sx} \text{ if } sz = zs^* < z, sx < x; \\
N_{szs^*}^x &= N_z^{sx} + (u^2 - 1) N_z^x \text{ if } sz \neq zs^* > z, sx < x; \\
u^2 N_{szs^*}^x &= N_z^{sx} \text{ if } sz \neq zs^* < z, sx < x; \\
u N_z^x + (1 + u) N_{sz}^x &= u^2 N_z^{sx} \text{ if } sz = zs^* > z, sx > x; \\
(u^2 - u) N_{sz}^x + (u^2 - u - 1) N_z^x &= u^2 N_z^{sx} \text{ if } sz = zs^* < z, sx > x; \\
N_{szs^*}^x &= u^2 N_z^{sx} \text{ if } sz \neq zs^* > z, sx > x; \\
u^2 N_{szs^*}^x + (u^2 - 1) N_z^x &= u^2 N_z^{sx} \text{ if } sz \neq zs^* < z, sx > x.
\end{aligned}$$

Comparing with the formulas in 1.6, we see that the family $\{\tilde{L}_z^x; (x, z) \in W \times \mathbf{I}_*\}$ is admissible. Hence there is a unique \mathfrak{H} -linear map $\mu : M \rightarrow \hat{\mathfrak{H}}$ such that for any $z \in \mathbf{I}_*$ we have $\mu(a_z) = \sum_{x \in W} \tilde{L}_z^x T_x$. Since $L_1^x = \delta_{x, x^*} u^{l(x)}$ (see [L6, 1.8]), we have $\tilde{L}_1^x = \delta_{x, x^*} (-1)^{l(x)} (-u)^{-l(x)} = \delta_{x, x^*} u^{-l(x)}$, so that $\mu(a_1) = X$ (see 0.1). Thus the existence part in 0.2(a) is established. The uniqueness part in 0.2(a) is obvious since a_1 generates M as an \mathfrak{H} -module. Since $L_z^x \in \mathbf{Z}[u]$ we see that $\tilde{L}_z^x \in \mathbf{Z}[u^{-1}]$ and 0.2(b) is established.

1.8. The algebra \mathfrak{H} and its module M can be specialized to $u = 0$. Then \mathfrak{H} becomes a \mathbf{Q} -algebra \mathfrak{H}_0 with basis $\{\underline{T}_w; w \in W\}$ and multiplication given by $\underline{T}_w \underline{T}_{w'} = \underline{T}_{ww'}$ if $l(ww') = l(w) + l(w')$, $(\underline{T}_s + 1) \underline{T}_s = 0$ if $s \in S$; M becomes a \mathbf{Q} -vector space M_0 with basis $\{\underline{a}_w; w \in \mathbf{I}_*\}$ and with \mathfrak{H}_0 -module structure given by

$$\begin{aligned}
\underline{T}_s \underline{a}_w &= \underline{a}_{sw} \text{ if } sw = ws^* > w; \\
\underline{T}_s \underline{a}_w &= \underline{a}_{sws^*} \text{ if } sw \neq ws^* > w; \\
\underline{T}_s \underline{a}_w &= -\underline{a}_w \text{ if } sw < w.
\end{aligned}$$

Here $s \in S, w \in \mathbf{I}_*$. We have the following result:

(a) *There is a unique map $W \times \mathbf{I}_* \rightarrow \mathbf{I}_*$, $(x, w) \mapsto x \circ w$ such that $\underline{T}_x \underline{a}_w = \epsilon_{x, w} \underline{a}_{x \circ w}$ for any $x \in W, w \in \mathbf{I}_*$; here $\epsilon_{x, w} = \pm 1$ is a well defined sign.*

We argue by induction on $l(x)$. If $x = 1$ we have $\underline{T}_x \underline{a}_w = \underline{a}_w$ so that $1 \circ w = w, \epsilon_{1, w} = 1$. Assume now that $x \neq 1$. Let $s \in S, x' \in W$ be such that $x = sx' > x'$. By the induction hypothesis we have $\underline{T}_{x'} \underline{a}_w = \pm \underline{a}_{u'}$ for some $u' \in \mathbf{I}_*$. Hence $\underline{T}_x \underline{a}_w = \pm \underline{T}_s \underline{a}_{u'}$ and this equals $\pm \underline{a}_u$ for some $u \in \mathbf{I}_*$. This proves (a).

We show:

(b) $\underline{T}_x \underline{a}_1 = (-1)^{l(x)} \epsilon(x \circ 1) \underline{a}_{x \circ 1}$ for any $x \in W$.

We argue by induction on $l(x)$. If $x = 1$ we have $\underline{T}_x \underline{a}_1 = \underline{a}_1$ hence $1 \circ 1 = 1$ and $l(x) = 0$, $\epsilon(1 \circ 1) = \epsilon(1) = 1$ and the result holds. Assume now that $x \neq 1$. Let $s \in S$, $x' \in W$ be such that $x = sx' > x'$. By the induction hypothesis we have $\underline{T}_{x'} \underline{a}_1 = (-1)^{l(x')} \epsilon(w) \underline{a}_w$ where $w = x' \circ 1$. We have $\underline{T}_x \underline{a}_1 = \underline{T}_s \underline{T}_{x'} \underline{a}_1 = (-1)^{l(x')} \epsilon(w) \underline{T}_s \underline{a}_w$. Now $\underline{T}_s \underline{a}_w = f(s, w) \underline{a}_{s \circ w}$ where $f(s, w) = 1$ if $sw > w$, $f(s, w) = -1$ if $sw < w$. It is enough to prove that $(-1)^{l(x)} \epsilon(x \circ 1) = (-1)^{l(x')} \epsilon(w) f(s, w)$. Since $l(x) = l(x') + 1$ it is enough to prove that $\epsilon(x \circ 1) = -\epsilon(w) f(s, w)$. We have $x \circ 1 = s \circ w$ hence it is enough to prove that $\epsilon(s \circ w) = -\epsilon(w) f(s, w)$ or that: $\epsilon(sw) = -\epsilon(w)$ if $sw = ws^* > w$, $\epsilon(sw) = -\epsilon(w)$ if $sw \neq ws^* > w$. This is clear from the definition of ϵ . This proves (b).

We define $\pi : W \rightarrow \mathbf{I}_*$ by $\pi(x) = x \circ 1$. We show:

(c) π is surjective.

Let $w \in \mathbf{I}_*$. We show by induction on $l(w)$ that $w \in \pi(W)$. If $w = 1$ we have $w = \pi(1)$. Assume now that $w \neq 1$. We can find $s \in S$ such that $sw < w$. Assume first that $sw = ws^*$. Then by the induction hypothesis we have $sw = x \circ 1$ for some $x \in W$ hence $\underline{T}_s \underline{T}_x \underline{a}_1 = \pm \underline{T}_s \underline{a}_{sw} = \pm \underline{a}_w$; moreover $\underline{T}_s \underline{T}_x$ equals \underline{T}_x (if $sx > x$) or \underline{T}_{sx} (if $sx < x$). Thus $w = x \circ 1$ or $w = sx \circ 1$.

Assume next that $sw \neq ws^*$. Then by the induction hypothesis we have $sws^* = x \circ 1$ for some $x \in W$ hence $\underline{T}_s \underline{T}_x \underline{a}_1 = \pm \underline{T}_s \underline{a}_{sws^*} = \pm \underline{a}_w$; moreover $\underline{T}_s \underline{T}_x$ equals \underline{T}_x (if $sx > x$) or \underline{T}_{sx} (if $sx < x$). Thus $w = x \circ 1$ or $w = sx \circ 1$. This proves (c).

1.9. For $x \in W$ we have $\underline{T}_x \underline{a}_1 = \sum_{z \in \mathbf{I}_*} \underline{L}_z^x \underline{a}_z$ where $\underline{L}_z^x = L_z^x|_{u=0} \in \mathbf{Z}$. Comparing with 1.8(b) we see that $\underline{L}_z^x = (-1)^{l(x)} \epsilon(z)$ if $z = x \circ 1$ and $\underline{L}_z^x = 0$ if $z \neq x \circ 1$. It follows that $\tilde{L}_z^x|_{u^{-1}=0} = 1$ if $z = x \circ 1$ and $\tilde{L}_z^x|_{u^{-1}=0} = 0$ if $z \neq x \circ 1$. Thus 0.2(c) holds.

1.10. We show that the map $\mu : M \rightarrow \hat{\mathfrak{H}}$ is injective. It is enough to show that the elements $\{\mu(a_z); z \in \mathbf{I}_*\}$ are linearly independent. Assume that $\sum_{z \in \mathbf{I}_*} \xi_z \mu(a_z) = 0$ where $\xi_z \in \mathbf{Q}(u)$ are zero for all but finitely many z and $\xi_z \neq 0$ for some $z \in \mathbf{I}_*$. We can assume that $\xi_z \in \mathbf{Z}[u^{-1}]$ for all z and $\xi_z|_{u^{-1}=0} \neq 0$ for some $z = z_0$. We have $\sum_{z \in \mathbf{I}_*, x \in W} \xi_z \tilde{L}_z^x T_x = 0$ hence $\sum_{z \in \mathbf{I}_*} \xi_z \tilde{L}_z^x = 0$ for any $x \in W$. Setting $u^{-1} = 0$ we deduce that $\sum_{z \in \mathbf{I}_*} \xi_z|_{u^{-1}=0} n_z^x = 0$ for any $x \in W$. By 0.2(c) this can be written as $\xi_{\pi(x)}|_{u^{-1}=0} = 0$ for any $x \in W$. By 1.8(c) we can find $x \in W$ such that $\pi(x) = z_0$. For this x we have $\xi_{z_0}|_{u^{-1}=0} = 0$. This is a contradiction, Thus the injectivity of μ is proved. This completes the proof of Theorem 0.2.

1.11. In the case where W is of type A_1 with $S = \{s\}$ we have $\mu(a_1) = u^{-1}T_s + 1$, $\mu(a_s) = (u-1)u^{-1}T_s$.

In the case where W is of type A_2 with $S = \{s, t\}$ and $* = 1$ we have

$$\mu(a_1) = u^{-3}T_{sts} + u^{-2}T_{st} + u^{-2}T_{ts} + u^{-1}T_s + u^{-1}T_t + 1,$$

$$\mu(a_s) = (u-1)(u^{-3}T_{sts} + u^{-2}T_{st} + u^{-1}T_s),$$

$$\mu(a_t) = (u-1)(u^{-3}T_{sts} + u^{-2}T_{ts} + u^{-1}T_t),$$

$\mu(a_{sts}) = (u-1)((u^{-1} + u^{-2} - u^{-3})T_{sts} + u^{-1}T_{st} + u^{-1}T_{ts}).$
(See [L4, 32, 3.3].)

1.12. For $x \in W, z \in \mathbf{I}_*$ we set $\tilde{L}_z^x = (u-1)^{\phi(z)}\lambda_z^x$ where $\phi(z)$ is as in 1.5 and $\lambda_z^x \in \mathbf{Q}(u)$. We show:

(a) $\lambda_z^x \in \mathbf{Z}[u^{-1}]$ and $\overline{\lambda_z^x} = (-u^2)^{l(x)+(1/2)(l(z)-\phi(z))}\lambda_z^x$.

From the definitions we have $L_z^1 = \delta_{1,z}$ hence $\tilde{L}_z^1 = \delta_{1,z}$ and $\lambda_z^1 = \delta_{1,z}$. From the formulas in 1.6 (with $s \in S$) we deduce (assuming $sz < x$):

$$\begin{aligned} \lambda_z^x &= \lambda_{sz}^x - u^{-1}\lambda_z^{sx} \text{ if } sz = zs^* < z; \\ \lambda_z^x &= u^{-1}\lambda_z^{sx} + (1 - u^{-2})\lambda_{sz}^{sx} \text{ if } sz = zs^* > z; \\ \lambda_z^x &= u^{-2}\lambda_{szs^*}^{sx} \text{ if } sz \neq zs^* > z; \\ \lambda_z^x &= \lambda_{szs^*}^{sx} + (1 - u^{-2})\lambda_z^{sx} \text{ if } sz \neq zs^* < z. \end{aligned}$$

From this (a) follows by induction on $l(x)$.

1.13. In this subsection we give an application of the function $\epsilon : \mathbf{I}_* \rightarrow \{\pm 1\}$ in 1.5. Let $E = \mathbf{Q}(u)$ viewed as an \mathfrak{H} -module in which T_x ($x \in W$) acts as multiplication by $(-1)^{l(x)}$ (sign representation of \mathfrak{H}). We define a $\mathbf{Q}(u)$ -linear map $f : M \rightarrow E$ by $f(a_z) = \epsilon(z)$. We claim that f is \mathfrak{H} -linear. It is enough to show that for any $w \in \mathbf{I}_*, s \in S$ we have:

$$\begin{aligned} -\epsilon(w) &= u\epsilon(w) + (u+1)\epsilon(sw) \text{ if } sw = ws^* > w; \\ -\epsilon(w) &= (u^2 - u - 1)\epsilon(w) + (u^2 - u)\epsilon(sw) \text{ if } sw = ws^* < w; \\ -\epsilon(w) &= \epsilon(sws^*) \text{ if } sw \neq ws^* > w; \\ -\epsilon(w) &= (u^2 - 1)\epsilon(w) + u^2\epsilon(sws^*) \text{ if } sw \neq ws^* < w. \end{aligned}$$

This follows from 1.5(a).

2. THE BIREGULAR REPRESENTATION OF \mathfrak{H}

2.1. In this section we discuss the special case of Theorem 0.2 in the case where W in 0.1 is replaced by $W^2 = W \times W$, S is replaced by $S^2 = S \times \{1\} \cup \{1\} \times S$ and $*$: $W \rightarrow W$ is replaced by $*$: $W^2 \rightarrow W^2$, $(x, y) \mapsto (y, x)$. In this case we have $\mathbf{I}_* = \{(x, y) \in W^2; xy = 1\}$.

We use the notation $\mathfrak{H}, \hat{\mathfrak{H}}$ in reference to W . Let $\mathfrak{H}^{(2)}, \hat{\mathfrak{H}}^{(2)}$ be the objects analogous to $\mathfrak{H}, \hat{\mathfrak{H}}$ defined in terms of W^2 instead of W . Thus $\mathfrak{H}^{(2)} = \mathfrak{H} \otimes \mathfrak{H}$ with basis $\{T_w \otimes T_{w'}; (w, w') \in W^2\}$ which is the analogue for $\mathfrak{H}^{(2)}$ of the basis $\{T_w; w \in W\}$ of \mathfrak{H} . We write T_w (resp. T'_w) instead of $T_w \otimes 1$ (resp. $1 \otimes T_w$). Then the basis element $T_w \otimes T_{w'}$ is actually the product in $\mathfrak{H}^{(2)}$ of T_w and $T'_{w'}$ in either order. In our case we have $M = \mathfrak{H}$ with the basis $\{a_{w, w^{-1}} = T_w; w \in W\}$ viewed as an $\mathfrak{H}^{(2)}$ -module in which the action of $T_x \otimes T_y$ is $T_r \mapsto T_x T_r T_{y^{-1}}$. (We refer to this as the *biregular representation*.) We can view $\mathfrak{H}^{(2)}$ as a subspace of $\hat{\mathfrak{H}}^{(2)}$ in an obvious way. The $\mathfrak{H}^{(2)}$ -module structure on $\mathfrak{H}^{(2)}$ (left multiplication) extends in an obvious way to a $\mathfrak{H}^{(2)}$ -module structure on $\hat{\mathfrak{H}}^{(2)}$. The following is a restatement of Theorem 0.2 in our case.

Corollary 2.2. (a) *There exists a unique $\mathfrak{H}^{(2)}$ -linear map $\mu : \mathfrak{H} \rightarrow \hat{\mathfrak{H}}^{(2)}$ (where $\mathfrak{H}, \hat{\mathfrak{H}}^{(2)}$ are viewed as $\mathfrak{H}^{(2)}$ -modules as above) such that $\mu(1) = \sum_{w \in W} u^{-2l(w)} T_w \otimes$*

$T_w \in \hat{\mathfrak{H}}^{(2)}$. Moreover, μ is an isomorphism of \mathfrak{H} onto the $\mathfrak{H}^{(2)}$ -submodule of $\hat{\mathfrak{H}}^{(2)}$ generated by $\mu(1)$.

(b) Let $z \in W$; we set $\mu(T_z) = \sum_{(x,y) \in W^2} N_z^{x,y} T_x \otimes T_y$ where $N_z^{x,y} \in \mathbf{Q}(u)$. For any $(x,y) \in W^2$ we have $N_z^{x,y} \in \mathbf{Z}[u^{-1}]$, hence we can define $n_z^{x,y} = N_z^{x,y}|_{u^{-1}=0} \in \mathbf{Z}$.

(c) There is a unique surjective function $\pi : W^2 \rightarrow W$ such that for $(x,y) \in W$, $z \in W$ we have $n_z^{x,y} = 1$ if $z = \pi(x,y)$, $n_z^{x,y} = 0$ if $z \neq \pi(x,y)$. (Note that $\pi(1,1) = 1$.)

In the remainder of this section we shall indicate a proof of a part of Corollary which is somewhat different from that of Theorem 0.2.

2.3. Let $\tau : \mathfrak{H} \rightarrow \mathbf{Q}(u)$ be the $\mathbf{Q}(u)$ -linear map such that $\tau(T_x) = 0$ if $x \neq 1$, $\tau(T_1) = 1$. For $x, y \in W$ we have $\tau(T_x T_y) = 0$ if $xy \neq 1$, $\tau(T_x T_y) = u^{2l(x)}$ if $xy = 1$. (See [L3, 10.4(a)].) It follows that for x, y in W we have

$$T_x T_y = \sum_{z \in W} \tau(T_x T_y T_z) u^{-2l(z)} T_{z^{-1}}.$$

Since $T_x \mapsto T_{x^{-1}}$ defines an antiautomorphism of \mathfrak{H} we have

$$T_{y^{-1}, x^{-1}} = \sum_{z \in W} \tau(T_x T_y T_z) u^{-2l(z)} T_z = \sum_{z \in W} \tau(T_{y^{-1}} T_{x^{-1}} T_{z^{-1}}) u^{-2l(z)} T_z$$

hence

$$(a) \quad \tau(T_x T_y T_z) = \tau(T_{y^{-1}} T_{x^{-1}} T_{z^{-1}})$$

for any x, y, z in W . By [L3, 10.4(b)] we have $\tau(hh') = \tau(h'h)$ for any h, h' in \mathfrak{H} . In particular for x, y, z in W we have

$$(b) \quad \tau(T_x T_y T_z) = \tau(T_y T_z T_x) = \tau(T_z T_x T_y).$$

Lemma 2.4. (a) For x, y, z in W we set $p_{x,y,z} = \tau(T_x T_y T_z) u^{-2l(z)-2l(y)}$, $p'_{x,y,z} = \tau(T_x T_y T_z) u^{-2l(x)}$. We have $p_{x,y,z} \in d_{x,y,z} + u^{-1} \mathbf{Z}[u^{-1}]$, $p'_{x,y,z} \in d'_{x,y,z} + u \mathbf{Z}[u]$ where $d_{x,y,z} \in \{0, 1\}$, $d'_{x,y,z} \in \{0, \pm 1\}$. Moreover $p'_{x,y,z} = (-1)^{l(x)+l(y)+l(z)} \overline{p_{x,y,z}}$ hence $d'_{x,y,z} = (-1)^{l(x)+l(y)+l(z)} d_{x,y,z}$.

(b) Let y, z be in W . There is exactly one $x \in W$ (denoted by $y * z$) such that $d_{x,y,z} = 1$ (or equivalently such that $d'_{x,y,z} = \pm 1$). For all other x we have $d_{x,y,z} = d'_{x,y,z} = 0$.

We argue by induction on $l(z)$. If $z = 1$ we have $p_{x,y,z} = \tau(T_x T_y) u^{-2l(y)} = \delta_{x,y}$, $p'_{x,y,z} = \tau(T_x T_y) u^{-2l(x)} = \delta_{x,y}$. Hence (a), (b) hold with $y * z = y^{-1}$. (We have $\delta_{x,y} = (-1)^{l(x)+l(y)} \delta_{x,y}$.)

Assume now that $l(z) \geq 1$. We write $z = sz'$, $s \in S$, $l(z') = l(z) - 1$. If $ys > y$ we have by the induction hypothesis

$$p_{x,y,z} = \tau(T_x T_y T_s T_{z'}) u^{-2l(z')-2l(y)-2} = p_{x,ys,z'} \in d_{x,ys,z'} + u^{-1} \mathbf{Z}[u^{-1}],$$

$$p'_{x,y,z} = \tau(T_x T_y T_s T_{z'}) u^{-2l(x)} = p'_{x,ys,z'} \in d'_{x,ys,z'} + u \mathbf{Z}[u]$$

hence the result holds: we have $d_{x,y,z} = d_{x,ys,z'}$, $d'_{x,y,z} = d'_{x,ys,z'}$, $y * z = (ys) * (sz)$. (We use that $(-1)^{l(x)+l(y)+l(z)} = (-1)^{l(x)+l(ys)+l(z')}$.)

If $ys < y$ we have by the induction hypothesis

$$\begin{aligned} p_{x,y,z} &= \tau(T_x T_y T_s T_{z'}) u^{-2l(z')-2l(y)-2} \\ &= p_{x,ys,z'} u^{-2} + p_{x,y,z'} u^{-2}(u^2 - 1) \in d_{x,y,z'} + u^{-1} \mathbf{Z}[u^{-1}], \end{aligned}$$

$$\begin{aligned} p'_{x,y,z} &= \tau(T_x T_y T_s T_{z'}) u^{-2l(x)} \\ &= p'_{x,ys,z'} u^2 + p'_{x,y,z'} (u^2 - 1) \in -d'_{x,y,z'} + u \mathbf{Z}[u] \end{aligned}$$

hence the result holds: we have $d_{x,y,z} = d_{x,y,z'}$, $d'_{x,y,z} = -d'_{x,y,z'}$, $y * z = y * (sz)$.

(We use that

$$\frac{(-1)^{l(x)+l(y)+l(z)}}{u^2 - 1} = (-1)^{l(x)+l(y)+l(z)}, \quad (-1)^{l(x)+l(y)+l(z)} = -(-1)^{l(x)+l(y)+l(z')},$$

2.5. For $a \in W$ we show

$$(a) \quad T_a X = T'_{a^{-1}} X.$$

We have

$$\begin{aligned} T_a X &= \sum_{w,z \in W} u^{-2l(w)-2l(z)} \tau(T_a T_w T_z^{-1}) T_z T'_w \in \hat{\mathfrak{H}}^{(2)}, \\ T'_{a^{-1}} X &= \sum_{w,z \in W} u^{-2l(w)-2l(z)} \tau(T_{a^{-1}} T_w T_{z^{-1}}) T_w T'_z \in \hat{\mathfrak{H}}^{(2)}. \end{aligned}$$

Making the change of variable $(w, z) \mapsto (z, w)$ in the last sum we obtain

$$T'_{a^{-1}} X = \sum_{w,z \in W} u^{-2l(w)-2l(z)} \tau(T_{a^{-1}} T_z T_{w^{-1}}) T_z T'_w.$$

It remains to show:

$$\tau(T_{a^{-1}} T_z T_{w^{-1}}) = \tau(T_a T_w T_{z^{-1}}).$$

Indeed, by 2.3(a) the left hand side is equal to $\tau(T_{z^{-1}} T_a T_w)$ and by 2.3(b) this is equal to the right hand side.

2.6. We give an alternative proof of the existence of μ in Corollary 2.2. For any $a \in W$ we set $X_a = T_a X = T'_{a^{-1}} X \in \hat{\mathfrak{H}}^{(2)}$, see 2.5(a). Thus, $X_1 = X$. We define a $\mathbf{Q}(u)$ -linear map $\mu : \mathfrak{H} \rightarrow \hat{\mathfrak{H}}^{(2)}$ by $T_a \mapsto X_a$ for all $a \in W$. For $h \in \mathfrak{H}$, $r \in W$ we have $\mu(T_r h) = T_r \mu(h)$ (using the description $X_a = T'_{a^{-1}} X$) and $\mu(h T_{r^{-1}}) = T'_r \mu(h)$ (using the description $X_a = T_a X$). It follows that μ is $\mathfrak{H}^{(2)}$ -linear.

In our case $\pi : W^2 \rightarrow W$ is given by $\pi(x, y) = (y * (x^{-1}), (y * (x^{-1}))^{-1})$.

2.7. In the case where W is of type A_1 with $S = \{s\}$ we have

$$\begin{aligned} \mu(T_1) &= T_1 \otimes T_1 + u^{-2} T_s \otimes T_s, \\ \mu(T_s) &= T_1 \otimes T_s + T_s \otimes T_1 + (1 - u^{-2}) T_s \otimes T_s. \end{aligned}$$

3. Γ -EQUIVARIANT VECTOR BUNDLES ON Γ

3.1. Let Γ be a finite group. Let $K_\Gamma(\Gamma)$ be the Grothendieck group of Γ -equivariant (complex) vector bundles on Γ where Γ acts on Γ by conjugation. For $x \in \Gamma$ let $\Gamma_x = Z_\Gamma(x)$ and let $\text{Irr}\Gamma_x$ be a set of representatives for the isomorphism classes of irreducible representations of Γ_x over \mathbf{C} . For any $x \in \Gamma$ and any $\rho \in \text{Irr}\Gamma_x$ there is a unique (up to isomorphism) Γ -equivariant vector bundle $E_{x,\rho}$ on Γ such that the support of $E_{x,\rho}$ is the conjugacy class of x and is such that the action of Γ_x on the fibre of $E_{x,\rho}$ is isomorphic to ρ . Let $\underline{\Gamma}$ be a set of representatives for the conjugacy classes in Γ . Let $\mathfrak{M}(\Gamma) = \{(x, \rho); x \in \underline{\Gamma}, \rho \in \text{Irr}\Gamma_x\}$. The classes of $E_{x,\rho}$ (with $(x, \rho) \in \mathfrak{M}(\Gamma)$) form a \mathbf{Z} -basis of $K_\Gamma(\Gamma)$.

Following Kottwitz [Ko] we consider the element $\kappa \in K_\Gamma(\Gamma)$ defined by

$$\kappa = \sum_{(x,\rho) \in \mathfrak{M}(\Gamma)} \sum_{s \in \Gamma; s^2=x} \frac{|Z_{\Gamma_x}|}{|\Gamma_x|} (1 : \rho|_{Z_{\Gamma_x}(s)}) E_{x,\rho}$$

where $(1 : \rho|_{Z_{\Gamma_x}(s)})$ denotes the multiplicity of the unit representation of $Z_{\Gamma_x}(s)$.

Proposition 3.2. Define $V = \alpha! \mathbf{C}$ where $\alpha : \Gamma \rightarrow \Gamma$ is $g \mapsto g^2$. Note that V is a Γ -equivariant vector bundle on Γ . We have $V = \kappa$ in $K_\Gamma(\Gamma)$.

Let $\Gamma^{(2)} = \{(g, h) \in \Gamma \times \Gamma; gh = hg\}$. For any Γ -equivariant vector bundle \mathcal{V} on Γ we define $\phi_{\mathcal{V}} : \Gamma^{(2)} \rightarrow \mathbf{C}$ as follows: $\phi_{\mathcal{V}}(g, h)$ is trace of the action of h on the fibre of \mathcal{V} at g . For example, if $(x, \rho) \in \mathfrak{M}(\Gamma)$, we have

$$\phi_{E_{x,\rho}}(g, h) = |\Gamma_x|^{-1} \sum_{a \in \Gamma; aga^{-1}=x} \text{tr}(aha^{-1}, \rho).$$

Note that $\mathcal{V} \mapsto \phi_{\mathcal{V}}$ induces an injective linear map from the vector space $\mathbf{C} \otimes K_\Gamma(\Gamma)$ into the vector spaces of functions $\Gamma^{(2)} \rightarrow \mathbf{C}$, see [L2]. Hence it suffices to show

that $\phi_V = \phi_\kappa$. For $(g, h) \in \Gamma^{(2)}$ we have

$$\begin{aligned} \phi_\kappa(g, h) &= \sum_{x \in \Gamma, \rho \in \text{Irr}\Gamma_x} \frac{|\Gamma_x|}{|\Gamma|} \sum_{s \in \Gamma; s^2 = \xi} \frac{|Z_{\Gamma_x}(s)|}{|\Gamma_x|} (1 : \rho|_{\Gamma_x \cap \Gamma_s}) \phi_{E_{x, \rho}}(g, h) \\ &= \sum_{x \in \Gamma, \rho \in \text{Irr}\Gamma_x} |\Gamma|^{-1} \sum_{s \in \Gamma; s^2 = x} \sum_{u \in \Gamma_x \cap \Gamma_s} \text{tr}(u^{-1}, \rho) |\Gamma_x|^{-1} \sum_{a \in \Gamma; aga^{-1} = x} \text{tr}(aha^{-1}, \rho) \\ &= \sum_{x \in \Gamma} |\Gamma|^{-1} \sum_{s \in \Gamma; s^2 = x} \sum_{u \in \Gamma_x \cap \Gamma_s} |\Gamma_x|^{-1} \sum_{a \in \Gamma; aga^{-1} = x} |\{z \in G_x; zaha^{-1}z^{-1} = u^{-1}\}| \end{aligned}$$

Setting $s' = a^{-1}sa$, $u' = a^{-1}ua$, $z' = a^{-1}za$ we obtain

$$\begin{aligned} \phi_\kappa(g, h) &= \sum_{s' \in \Gamma; s'^2 = g} \sum_{u' \in \Gamma_g \cap \Gamma_{s'}} |\Gamma_g|^{-1} |\{z' \in G_g; z'hz'^{-1} = u'^{-1}\}| \\ &= \sum_{s' \in \Gamma; s'^2 = g} |\Gamma_g|^{-1} |\{z' \in G_g; z'hz'^{-1} \in G_g \cap G_{s'}\}|. \end{aligned}$$

Setting $\tilde{s} = z'^{-1}s'z'$ we obtain

$$\phi_\kappa(g, h) = |\{\tilde{s} \in \Gamma; \tilde{s}^2 = g, \tilde{s}h = h\tilde{s}\}|.$$

From the definitions we have

$$\phi_V(g, h) = |\{\tilde{s} \in \Gamma; \tilde{s}^2 = g, \tilde{s}h = h\tilde{s}\}|.$$

The proposition is proved.

3.3. As in [L2, 2.5], any $(y, \sigma) \in \mathfrak{M}(\Gamma)$ defines a \mathbf{C} -linear function $\chi_{y, \sigma} : \mathbf{C} \otimes K_\Gamma(\Gamma) \rightarrow \mathbf{C}$ by the rule

$$\chi_{y, \sigma}(U) = (\dim \sigma)^{-1} \sum_{\gamma \in G_y} \text{tr}(y, U_\gamma) \text{tr}(\gamma, \sigma)$$

for any Γ -equivariant vector bundle. (This is in fact an algebra homomorphism for the algebra structure defined in [L2, 2.2].) Moreover, if $(x, \rho) \in M(\Gamma)$ then

$$(a) \quad \chi_{y, \sigma}(E_{x, \rho}) = \frac{|\Gamma_y|}{\dim \sigma} \{(x, \rho), (y, \sigma^*)\}$$

where $\{, \}$ is the nonabelian Fourier transform matrix of [L1] and $\sigma^* \in \text{Irr}\Gamma_y$ is isomorphic to the dual of σ . We compute $\chi_{y, \sigma}(V)$ where V is as in 3.2 and σ has Frobenius-Schur indicator 1. By the proof of 3.2 we have

$$\begin{aligned} \chi_{y, \sigma}(V) &= (\dim \sigma)^{-1} \sum_{\gamma \in \Gamma_y} \text{tr}(y, V_\gamma) \text{tr}(\gamma, \sigma) \\ &= (\dim \sigma)^{-1} \sum_{\gamma \in \Gamma_y, \tilde{s} \in \Gamma_y; \tilde{s}^2 = \gamma} \text{tr}(\gamma, \sigma) = (\dim \sigma)^{-1} \sum_{\tilde{s} \in \Gamma_y} \text{tr}(\tilde{s}^2, \sigma) = \frac{|\Gamma_y|}{\dim \sigma}. \end{aligned}$$

Combining this with (a) we see that

$$(b) \quad \sum_{(x, \rho)} \{(x, \rho), (y, \sigma)\} \text{mult. of } E_{x, \rho} \text{ in } V = 1.$$

4. SOME APPLICATIONS OF THEOREM 0.2

4.1. Let A be a finite dimensional split semisimple algebra over a field K . Let $\text{Mod}A$ be the category of A -modules of finite dimension over K . For $E' \in \text{Mod}A$ let $A^{E'}$ be the sum of the simple two-sided ideals I of A such that $IE' \neq 0$. For $E, E' \in \text{Mod}A$ let $E_{E'} = A^{E'}E = \sum_{f \in \text{Hom}_A(E', E)} f(E')$. We have the following result.

(a) *Let $E \in \text{Mod}A$ and let $\mathcal{X} \in A$. We have a canonical K -linear isomorphism $\alpha : \mathcal{X}E \xrightarrow{\sim} \text{Hom}_A(A\mathcal{X}, E)$. Moreover, $\mathcal{X}E \subset E_{A\mathcal{X}}$. We have $A\mathcal{X}A \subset A^{A\mathcal{X}}$.* (Note that $A\mathcal{X}$ is a left ideal of A hence an object of $\text{Mod}A$.) For $e \in \mathcal{X}E$ we define $f_e : A\mathcal{X} \rightarrow E$ by $f_e(a\mathcal{X}) = ae, a \in A$; f_e is well defined: if $a, a' \in A$ satisfy $a\mathcal{X} = a'\mathcal{X}$ then $ae - a'e = a\mathcal{X}e_0 - a'\mathcal{X}e_0 = 0$ where $e = \mathcal{X}e_0, e_0 \in E$. Now $e \mapsto f_e$ is a K -linear map $\alpha : \mathcal{X}E \rightarrow \text{Hom}_A(A\mathcal{X}, E)$ which is clearly injective. We have $\dim_K(\mathcal{X}E) = \dim_K \text{Hom}_A(A\mathcal{X}, E)$. (We can assume that A is a simple K -algebra and E is a simple A -module. Thus we can assume that for some K -vector space V of finite dimension we have $A = \text{End}(V)$ and $E = V$ is viewed as an A -module in an obvious way. In this case the desired statement is easily verified.) It follows that α is an isomorphism.

We prove the second statement of (a). For $e \in \mathcal{X}E$ we have $f_e(\mathcal{X}) = e$. Since $f_e \in \text{Hom}_A(A\mathcal{X}, E)$ we see that $e \in E_{A\mathcal{X}}$, proving the second statement of (a). Applying this to $E = A$ viewed as an object of $\text{Mod}A$ under left multiplication we see that $\mathcal{X}A \subset A_{A\mathcal{X}}$. We now observe that $A_{A\mathcal{X}} \subset A^{A\mathcal{X}}$. Hence $\mathcal{X}A \subset A^{A\mathcal{X}}$ and $A\mathcal{X}A \subset AA^{A\mathcal{X}} = A^{A\mathcal{X}}$. This proves the third statement of (a).

In the remainder of this section we assume that W is a Weyl group and $* = 1$.

Theorem 4.2. *Let $M \in \text{Mod}\mathfrak{H}$, $X \in \mathfrak{H}$ be as in 0.1. Let $E \in \text{Mod}\mathfrak{H}$. We have canonically $XE \cong \text{Hom}_{\mathfrak{H}}(M, E)$. Moreover, $XE \subset E_M$ and $\mathfrak{H}X\mathfrak{H} \subset \mathfrak{H}^M$ (notation of 4.1).*

We apply 4.1 with $K = \mathbf{Q}(u)$, $A = \mathfrak{H} = \hat{\mathfrak{H}}$, $\mathcal{X} = X$ and we use Theorem 0.2. The theorem follows.

If E is a simple object of $\text{Mod}\mathfrak{H}$ then $\dim_K \text{Hom}_{\mathfrak{H}}(M, E)$ is known from the work of Kottwitz [Ko]; indeed, by [LV], the specialization of our M at $u = 1$ is (noncanonically) isomorphic to a W -module explicitly computed in [Ko]. In particular, using the theorem we see that (a),(b) below hold.

(a) If W is of type A_n and E is a simple \mathfrak{H} -module then $\dim_K(XE) = 1$; in particular, E contains a canonical line.

(b) If W is of type B_n or D_n and E is a simple \mathfrak{H} -module then $\dim_K(XE)$ is a power of 2 if E is a special representation (see [L1]) and $XE = 0$ if E is a nonspecial representation.

4.3. Let $\mathcal{A} = \mathbf{Z}[u, u^{-1}] \subset K$. Let \mathcal{H} be the \mathcal{A} -subalgebra of \mathfrak{H} with basis $\{T_w; w \in W\}$; \mathcal{H} is the same as the \mathcal{A} -algebra defined in [L3, 3.2] except that T_w, v of [L3, 3.2] are the same as $u^{-l(w)}T_w, u$ of this paper. (When we refer to [L3] we assume that $L = l$ as in [L3, 15.1].)

Let J be the asymptotic Hecke algebra (over \mathbf{Z}) with basis $\{t_z; z \in W\}$ associated to W , see [L3, §18]. Let $\mathbf{J} = \mathbf{Q} \otimes J$, ${}_K J = K \otimes \mathbf{J}$; these are split semisimple algebras.

Let $\{c_w; w \in W\}$ be the \mathcal{A} -basis of \mathcal{H} as in [L3, 5.2]. For x, y, z in W let $h_{x,y,z} \in \mathcal{A}$ be as in [L3, 13.1]. For x, y in W we write $x \sim y$ if x, y are in the same left cell. For $x \in W$ let $a(x) \in \mathbf{N}$ be as in [L3, 13.6]. Let $\mathcal{D} \subset W$ be as in [L3, 14.1]. The K -linear map $\psi : \mathfrak{H} \rightarrow {}_K \mathbf{J}$ given by $c_x \mapsto \sum_{d \in \mathcal{D}, z \in W; d \sim z^{-1}} h_{x,d,z} t_z$ is a K -algebra isomorphism (see [L3, 18.8]).

For any $\mathcal{E} \in \text{Mod} \mathbf{J}$ we set ${}_K \mathcal{E} = K \otimes \mathcal{E} \in \text{Mod}({}_K \mathbf{J})$; let \mathcal{E}_u be the \mathfrak{H} -module corresponding to ${}_K \mathcal{E}$ under ϕ . Let $\mathcal{M} \in \text{Mod}(\mathbf{J})$ be such that $\mathcal{M}_u \cong M$.

From 4.2 we deduce the following result.

Corollary 4.4. *Let $\mathcal{E} \in \text{Mod} \mathbf{J}$. We have*

$$\dim_K(\psi(X)({}_K \mathcal{E})) = \dim_K \text{Hom}_{\mathfrak{H}}(M, \mathcal{E}_u).$$

Moreover, $\psi(X)({}_K \mathcal{E}) \subset ({}_K \mathcal{E})_{K\mathcal{M}}$.

4.5. For x, y, z in W we have

(a) $h_{x,y,z} = \gamma_{x,y,z^{-1}} u^{a(z)} + \text{lower powers of } u$ where $\gamma_{x,y,z^{-1}} \in \mathbf{N}$, see [L3, 13.6]. For $x \in W$ we have $u^{-l(w)} T_w = \sum_{y \in W} s_{y,w} c_y$ where

(b) $s_{y,w} \in u^{-1} \mathbf{Z}[u^{-1}]$ for all $y \neq w$ and $s_{w,w} = 1$.

Proposition 4.6. *Let Z be a left cell of W and let $a = a(h)$ for any $h \in Z$. Let $\xi \in W$ be such that $\xi^{-1} \in Z$. We have*

$$\psi(X)t_\xi = \sum_{z \in Z} r_z t_z t_\xi$$

where $r_z = u^a + \sum_{i < a} n_{i,z} u^i$ and $n_{i,z} \in \mathbf{Z}$ are zero for all but finitely many i .

From the definitions we have

$$\begin{aligned} \psi(X)t_\xi &= \sum_{w \in W} \psi(u^{-l(w)} T_w) t_\xi = \sum_{y, w \in W} s_{y,w} \psi(c_y) t_\xi \\ &= \sum_{y, w, z \in W, d \in \mathcal{D}; a(d)=a(z)} s_{y,w} h_{y,d,z} t_z t_\xi. \end{aligned}$$

By [L3, 14.2], in the last sum we can assume that $z \in Z$ and that $d \in Z$. Hence

$$\psi(X)t_\xi = \sum_{y, w \in W, z \in Z, d \in \mathcal{D} \cap Z} s_{y,w} h_{y,d,z} t_z t_\xi.$$

Using 4.5(a),(b), we see that

$$\psi(X)t_\xi = \sum_{y \in W, z \in Z, d \in \mathcal{D} \cap Z} \gamma_{y,d,z^{-1}} u^a t_z t_\xi + \text{lower powers of } u.$$

Using [L3, 14.2], we see that $\gamma_{y,d,z^{-1}}$ is 1 if $y = z$ and is 0 otherwise. Thus we have

$$\psi(X)t_\xi = \sum_{z \in Z} u^a t_z t_\xi + \text{lower powers of } u.$$

The proposition is proved.

Corollary 4.7. *Let Z, Z' be two left cells of W such that $Z \cap Z'^{-1} \neq \emptyset$. We have $\sum_{z \in Z \cap Z'^{-1}} t_z \in \mathbf{J}^\mathcal{M}$.*

Let $a = a(w)$ for any $w \in Z$. Let d (resp. d') be the unique element of $\mathcal{D} \cap Z$ (resp. $\mathcal{D} \cap Z'$). From 4.2 we deduce (using ψ) that $\psi(X)t_d \in ({}_K\mathbf{J})^{\kappa\mathcal{M}} = K \otimes (\mathbf{J}^\mathcal{M})$. Using now 4.6 we deduce that $\sum_{z \in Z} (u^a + \sum_{i < a} n_{i,z} u^i) t_z \in K \otimes (\mathbf{J}^\mathcal{M})$. It follows that $\sum_{z \in Z} t_z \in \mathbf{J}^\mathcal{M}$ hence $t_{d'} \sum_{z \in Z} t_z \in \mathbf{J}^\mathcal{M}$. We now note that $t_{d'} \sum_{z \in Z} t_z = \sum_{z \in Z \cap Z'^{-1}} t_z$. The corollary is proved.

4.8. We now assume in addition that W is of type A_n, B_n or D_n . Then, by 4.2(a),(b), the two-sided ideal $\mathbf{J}^\mathcal{M}$ of \mathbf{J} is the sum of the simple two-sided ideals corresponding to the various special representations of W . The dimension of this sum is equal to number of pairs of left cells Z, Z' such that $Z \cap Z'^{-1}$. Hence in this case, from 4.7 we deduce:

(a) *The elements $\sum_{z \in Z \cap Z'^{-1}} t_z$ for various Z, Z' as above form a \mathbf{Q} -basis of $\mathbf{J}^\mathcal{M}$.*

It follows that for any two-sided cell c of W and any left cell Z contained in c ,

(b) *the elements $\sum_{z \in Z \cap Z'^{-1}} t_z$ (for various left cells Z' contained in c) form a \mathbf{Q} -basis of the unique left \mathbf{J} -submodule of $\oplus_{z \in Z} t_z$ isomorphic to the special representation of \mathbf{J} associated to c .*

4.9. For irreducible W of exceptional type, the elements described in 4.8(a) do not span the \mathbf{Q} -vector space $\mathbf{J}^\mathcal{M}$. For example, if W is of type G_2 , that is, a dihedral group with generators s_1, s_2 such that $(s_1 s_2)^6 = 1$, then (a) provides only 6 elements while $\dim \mathbf{J}^\mathcal{M} = 8$. If we write $t_{12\dots}$ instead of $t_{s_1 s_2 \dots}$, $t_{21\dots}$ instead of $t_{s_2 s_1 \dots}$ and t_\emptyset instead of $t_{\text{unit element}}$, then the following 8 elements form a \mathbf{Q} -basis of $\mathbf{J}^\mathcal{M}$:

$$(a) \quad t_\emptyset, t_1 + t_{12121}, t_{121}, t_2 + t_{21212}, t_{212}, t_{12} + t_{1212}, t_{21} + t_{2121}, t_{121212}.$$

This, together with 4.8(a), suggests that for any W , $\mathbf{J}^\mathcal{M}$ admits a \mathbf{Q} -basis consisting of \mathbf{N} -linear combinations of elements t_z .

4.10. Let $M_\mathcal{A}$ be the \mathcal{A} -submodule of M with basis $\{a_w; w \in \mathbf{I}_*\}$. Note that the \mathfrak{H} -module structure on M restricts to an \mathcal{H} -module structure on $M_\mathcal{A}$. For any $\lambda \in \mathbf{C}^*$ we regard \mathbf{C} as an \mathcal{A} -module via $u \mapsto \lambda$. We can then form $M_\lambda = \mathbf{C} \otimes_\mathcal{A} M_\mathcal{A}$, $\mathcal{H}_\lambda = \mathbf{C} \otimes_\mathcal{A} \mathcal{H}$ and M_λ becomes a module over the \mathbf{C} -algebra \mathcal{H}_λ . Let $X_\lambda = 1 \otimes X \in \mathcal{H}_\lambda$ where X is as in 0.1. Now the assignment $a_z \mapsto \sum_{x \in W} \tilde{L}_z^x T_x$ in 1.7 defines an \mathcal{H} -linear map $\mu_\mathcal{A} : M_\mathcal{A} \rightarrow \mathcal{H}$ such that $\mu_\mathcal{A}(a_1) = X$; by extension of scalars this gives rise to an \mathcal{H}_λ -linear map $\mu_\lambda : M_\lambda \rightarrow \mathcal{H}_\lambda$ such that $\mu_\lambda(a_1) = X_\lambda$.

Now, if $\lambda \neq -1$, the \mathcal{H}_λ -module M_λ is generated by a_1 ; it follows that in this case the image of μ_λ is the left ideal of \mathcal{H}_λ generated by X_λ . From Theorem 0.2 it follows that there exists a finite subset S_0 of \mathbf{C}^* such that $-1 \in S_0$ and such that

(a) for $\lambda \in \mathbf{C}^* - S_0$, $\mu_\lambda : M_\lambda \rightarrow \mathcal{H}_\lambda X_\lambda$ is an isomorphism of \mathcal{H}_λ -modules.
(Examples in small rank suggest that one can take $S_0 = \{1, -1\}$.)

4.11. We now assume that λ in 4.10 is such that $\lambda^2 = q$ where q is a power of a prime number. We write $\lambda = \sqrt{q}$. Let G be a split semisimple algebraic group defined over the finite field \mathbf{F}_q and let $G(\mathbf{F}_q)$ the (finite) group of \mathbf{F}_q -rational points of G . Let \mathcal{B} be the flag manifold of G and let $\mathcal{B}(\mathbf{F}_q)$ the set of \mathbf{F}_q -rational points of G . Let \mathcal{F} be the vector space of functions $\mathcal{B}(\mathbf{F}_q) \rightarrow \mathbf{C}$. For any $B \in \mathcal{B}(\mathbf{F}_q)$ let $f_B \in \mathcal{F}$ be the function defined by $f_B(B') = \sqrt{q}^{l(w)}$ for any $B' \in \mathcal{B}(\mathbf{F}_q)$ such that (B, B') are in relative position $w \in W$. Let \mathcal{F}' be the \mathbf{C} -subspace of \mathcal{F} spanned by the functions f_B for various $B \in \mathcal{B}(\mathbf{F}_q)$. Note that \mathcal{F} has a natural linear action of $G(\mathbf{F}_q)$ whose commuting algebra can be identified with $\mathcal{H}_{\sqrt{q}}$. Then \mathcal{F}' is a $G(\mathbf{F}_q)$ -invariant space of \mathcal{F} . Moreover we have $\mathcal{F}' = X_{\sqrt{q}}\mathcal{F}$. For each two-sided cell c of W we denote by \mathcal{F}_c (resp. \mathcal{F}'_c) the sum of all simple $G(\mathbf{F}_q)$ -submodules of \mathcal{F} (resp. \mathcal{F}') which belong to c in the classification of [L1]. Note that \mathcal{F}_c is an $\mathcal{H}_{\sqrt{q}}$ -submodule of \mathcal{F} and that $\mathcal{F}'_c = X_{\sqrt{q}}\mathcal{F}_c$. We have the following result.

Proposition 4.12. *Assume that $\sqrt{q} \notin S_0$. Let $a' = a(w_0 w)$ where w is any element of c and w_0 is the longest element of W .*

(a) *We have $\dim(\mathcal{F}'_c) = P_c(q)$ where $P_c \in \mathbf{N}[t]$ (t an indeterminate) is of the form $t^{a'} + \text{higher powers of } t$. Moreover, $P_c(1)$ is the number of involutions contained in c .*

(b) *We have $\dim(\mathcal{F}') = P(q)$ where $P \in \mathbf{N}[t]$ is such that $P(1)$ is the number of involutions in W .*

We prove (a). We can assume that W is irreducible. The simple $\mathcal{H}_{\sqrt{q}}$ -modules which belong to c can be indexed as in [L1] by a subset I of $M(\Gamma)$ (see 3.1) for a certain finite group Γ associated to c ; we write ϵ_i for the simple $\mathcal{H}_{\sqrt{q}}$ -module indexed by $i \in I$ and ρ_i for the corresponding simple $G(\mathbf{F}_q)$ -module appearing in \mathcal{F} .

We apply 4.1(a) with $A = \mathcal{H}_{\sqrt{q}}$, $\mathcal{X} = X_{\sqrt{q}}$, $E = \mathcal{F}_c$. We see that

$$\dim(\mathcal{F}'_c) = \dim \operatorname{Hom}_{\mathcal{H}_{\sqrt{q}}}(\mathcal{H}_{\sqrt{q}} X_{\sqrt{q}}, \mathcal{F}_c).$$

Using 4.10(a) we deduce

$$(c) \quad \dim(\mathcal{F}'_c) = \dim \operatorname{Hom}_{\mathcal{H}_{\sqrt{q}}}(M_{\sqrt{q}}, \mathcal{F}_c) = \sum_{i \in I} (\epsilon_i : M_{\sqrt{q}}) \dim \rho_i$$

where $(\epsilon_i : M_{\sqrt{q}})$ is the multiplicity of ϵ_i in $M_{\sqrt{q}}$. As explained in the remarks after Theorem 0.2, the multiplicity $(\epsilon_i : M_{\sqrt{q}})$ can be obtained from [Ko]; namely,

if $|I| = 2$, then $(\epsilon_i : M_{\sqrt{q}}) = 1$ for $i \in I$;

if $|I| \neq 2$, and $i = (x, \rho) \in I$, then $(\epsilon_i : M_{\sqrt{q}})$ is the multiplicity of $E_{x, \rho}$ in κ (see 3.1) or equivalently, the multiplicity of $E_{x, \rho}$ in V (see 3.2).

Thus, if $|I| = 2$ we have

$$\dim(\mathcal{F}'_c) = \sum_{i \in I} \dim \rho_i;$$

if $|I| \neq 2$ we have

$$(d) \quad \dim(\mathcal{F}'_c) = \sum_{(x, \rho) \in I} (\text{mult. of } E_{x, \rho} \text{ in } V) \dim \rho_{(x, \rho)}.$$

Let $d(\epsilon_i) \in \mathbf{N}[t]$ be the fake degree of ϵ_i . If $|I| = 2$ then by [L1] we have $\sum_{i \in I} \dim \rho_i = \sum_{i \in I} \delta(\epsilon_i)$. If $|I| \neq 2$ then by [L1] we have $\dim \rho_i = \sum_{i' \in I} \{i, i'\} d(\epsilon_{i'})$ where $\{i, i'\}$ is as in 3.3. Introducing this in (d) we obtain

$$\dim(\mathcal{F}'_c) = \sum_{(x, \rho) \in I} (\text{mult. of } E_{x, \rho} \text{ in } V) \sum_{(y, \sigma) \in I} \{(x, \rho), (y, \sigma)\} d(\epsilon_{y, \sigma})$$

Using now 3.3(b) we obtain

$$(e) \quad \dim(\mathcal{F}'_c) = \sum_{(y, \sigma) \in I} d(\epsilon_{y, \sigma}).$$

Here we have used the following two properties which are easily checked in each case.

(mult. of $E_{x, \rho}$ in V) $\neq 0 \implies (x, \rho) \in I$;

If $(y, \sigma) \in I$ then the Frobenius-Schur indicator of σ equals 1.

We see that (e) holds both when $|I| \neq 2$ and when $|I| = 2$. Now the first assertion of (a) follows immediately from (e); the second assertion of (a) also follows from (a) using the fact that $d(\epsilon_{y, s})|_{q=1} = \dim(\epsilon_{y, s})$ and that $\sum_{(y, \sigma) \in I} \dim(\epsilon_{y, \sigma})$ is equal to the number of involutions in c , see [Ge].

Clearly, (b) is a consequence of (a). The proposition is proved.

4.13. The proof of 4.12 shows that $\dim(\mathcal{F}')$ is equal to the sum of the fake degrees $d(\epsilon)$ of the various irreducible representations ϵ of $\mathcal{H}_{\sqrt{q}}$ (each one taken once).

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